Bipartite states of low rank are almost surely entangled

2009 J. Phys. A: Math. Theor. 42 095303

(http://iopscience.iop.org/1751-8121/42/9/095303)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 095303 (15pp) doi:10.1088/1751-8113/42/9/095303

Bipartite states of low rank are almost surely entangled

Mary Beth Ruskai[1,4] **and Elisabeth M Werner**[2,3,5]

[1] Department of Mathematics, Tufts University, Medford, MA 02155, USA
[2] Department of Mathematics, Case Western Reserve University Cleveland, OH 44106, USA
[3] Université de Lille 1, UFR de Mathématique, 59655 Villeneuve d'Ascq, France

E-mail: marybeth.ruskai@tufts.edu and elisabeth.werner@case.edu

Received 2 December 2008, in final form 9 January 2009
Published 5 February 2009
Online at stacks.iop.org/JPhysA/42/095303

Abstract
We show that a bipartite state on a tensor product of two matrix algebras is almost surely entangled if its rank is not greater than that of one of its reduced density matrices.

PACS numbers: 02.10.Yn, 03.67.Bg
Mathematics Subject Classification: 81P99, 81Q99

1. Introduction

1.1. Background

Recently, Arveson [2] considered the question of when a bipartite mixed state of rank r is almost surely entangled, and showed that this holds when $r \leqslant d/2$ where d is the dimension of the smaller space. In this paper, we show that this result holds if $r \leqslant d$, with d now the dimension of the larger space.

We will use results from [11] on entanglement breaking channels and exploit the well-known isomorphism between bipartite states and completely positive (CP) maps[6]. We will first consider states associated with completely positive trace-preserving (CPT) maps and then find that extension to arbitrary bipartite states is quite straightforward.

If the rank of a bipartite state γ_{AB} is strictly smaller than that of either of its reduced density matrices, then the state must be entangled. This is an immediate consequence of well-known results on entanglement, and seems to have first appeared explicitly in [12]. We

[4] Partially supported by National Science Foundation under grant DMS-0604900.
[5] Partially supported by an NSF grant, a FRG-NSF grant and a BSF grant.
[6] This isomorphism is usually attributed to Jamiolkowski [13] or to Choi [7], who used it to characterize the completely positive maps on finite-dimensional algebras. However, it seems to have been known to operator algebraists earlier and appeared implicitly in Arveson's proof of lemma 1.2.6 in [1].

include a proof in appendix A for completeness. This allows us to restrict attention to the case in which the ranks of the reduced density matrices are equal, with one of full rank.

Although it seems natural to expect that this result is optimal, recent results of Walgate and Scott [19] suggest otherwise. Let the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimensions d_A and d_B, respectively. It follows from a result proved independently by Wallach [20] and by Parthasarathy [15] for multi-partite entanglement that when $s > (d_A-1)(d_B-1)$ any subspace of $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimension s contains some product states, and that this bound is best possible, i.e., if $s \leqslant (d_A-1)(d_B-1)$ then there is some subspace of dimension s with no product states.

Walgate and Scott extended this by proving [19, corollary 3.5] that if a subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$ has dimension $s \leqslant (d_A-1)(d_B-1)$ then, almost surely, it contains no product states. For a bipartite state γ_{AB} with rank $r \leqslant (d_A-1)(d_B-1)$, it follows that the range of γ_{AB} almost surely contains no product states, which implies that a bipartite state γ_{AB} with rank $r \leqslant (d_A-1)(d_B-1)$ is almost surely entangled. Alternatively, one could apply [19, theorem 3.4] directly to $\ker(\gamma_{AB})$ to reach the same conclusion.

When $d_A > d_B \geqslant 2$, this result is stronger than ours, but for a pair of qubits, $d_A = d_B = 2$ our result is stronger. Moreover, it is easy to extend our qubit results to the general case of bipartite states with rank $r = d_A \geqslant d_B \geqslant 2$, providing a proof quite different from that in [19]. Although our measure is constructed differently from that used in [2], our approach is similar in the sense that we show that in a natural parameterization of the set of density matrices, the separable ones lie in a space of smaller dimension.

In the next half of this section, we review relevant terminology, and describe the notation and conventions we will use. Qubit channels and states are considered in section 2, and the general case in section 3. We conclude with some remarks about other approaches, and the question of the largest rank for which the separable states have measure zero.

1.2. Basics and notation

In this paper, we consider maps $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ and identify them with bipartite states or, equivalently, density matrices in $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ via the Choi–Jamiolkowski isomorphism as described below. Our primary interest is the situation in which $\mathcal{H}_A = \mathbf{C}_{d_A}$, in which case we can identify $\mathcal{B}(\mathbf{C}_d)$ with M_d, the space of $d \times d$ matrices. However, we will also have occasion to consider either Hilbert space $\mathcal{H}$ as a proper subspace of $\mathbf{C}_d$ for some d.

We will identify a state with a density matrix, i.e., a positive semi-definite operator ρ with $\mathrm{Tr}\,\rho = 1$, in $\mathcal{B}(\mathcal{H})$. To an operator algebraist this corresponds to the positive linear functional *on* the algebra $\mathcal{B}(\mathcal{H})$ which takes $A \mapsto \mathrm{Tr}\,\rho A$. In the physics and quantum information literature, a density matrix (or, more properly, a density operator) is often referred to as a (mixed) state *on* $\mathcal{H}$ (because the density operator acts on $\mathcal{H}$).

When $\mathcal{H}_A = \mathbf{C}_{d_A}$ and $\mathcal{H}_B = \mathbf{C}_{d_B}$, we write $\Phi : M_{d_A} \mapsto M_{d_B}$. In this case, let $\{e_j\}$ and $\{f_m\}$ denote orthonormal bases for $\mathbf{C}_{d_A}$ and $\mathbf{C}_{d_B}$, respectively. The isomorphism between states and matrices arises from the fact that

$$\mathrm{Tr}|f_m\rangle\langle f_n|\Phi(|e_j\rangle\langle e_k|) \tag{1}$$

can be interpreted as either

(i) the matrix representative of the linear map $\Phi : M_{d_A} \mapsto M_{d_B}$ in the bases $|f_m\rangle\langle f_n|$ and $|e_j\rangle\langle e_k|$ for M_{d_B} and M_{d_A} respectively, or,

(ii) the density matrix γ_{AB} of a state on $\mathbf{C}_{d_A} \otimes \mathbf{C}_{d_B}$ with elements $[\gamma_{AB}]_{jm,kn}$ in the product basis $|e_j \otimes f_m\rangle$.

Conversely, any state on $\mathbf{C}_{d_A} \otimes \mathbf{C}_{d_B}$ defines a CP map. We describe this well-known fact in detail in order to establish some conventions for interpretations of γ_A and γ_B. Observe that (ii) is equivalent to writing γ_{AB} as a block matrix of the form

$$\gamma_{AB} = \frac{1}{d_A} \sum_{jk} |e_j\rangle\langle e_k| \otimes P_{jk} = \frac{1}{d_A} \sum_{jk} |e_j\rangle\langle e_k| \otimes \Phi(|e_j\rangle\langle e_k|) \tag{2}$$

with the block $P_{jk} = \Phi(|e_j\rangle\langle e_k|)$ the matrix in M_{d_B} given by the image $\Phi(|e_j\rangle\langle e_k|)$. One can write an arbitrary matrix in $M_{d_A} \otimes M_{d_B}$ in the block form $\sum_{jk} |e_j\rangle\langle e_k| P_{jk}$ and then define $\Phi(|e_j\rangle\langle e_k|) = P_{jk}$ and extend by linearity or, equivalently,

$$\Phi(A) = \sum_{jk} a_{jk} P_{jk} \tag{3}$$

when $A = \sum_{jk} a_{jk} |e_j\rangle\langle e_k|$.

Observe that

$$\gamma_B = \frac{1}{d_A} \mathrm{Tr}_A \gamma_{AB} = \frac{1}{d_A} \sum_k \Phi(|e_k\rangle\langle e_k|) = \frac{1}{d_A} \Phi(I_A) \tag{4a}$$

$$\gamma_A = \frac{1}{d_A} \mathrm{Tr}_B \gamma_{AB} = \frac{1}{d_A} \sum_{jk} |e_j\rangle\langle e_k| \,\mathrm{Tr}\, \Phi(|e_j\rangle\langle e_k|) \tag{4b}$$

and that this implies the following:

(a) Φ is unital, i.e., $\Phi(I_A) = I_B$, if and only if $\gamma_B = \frac{1}{d_A} I_B$, and

(b) Φ is trace-preserving (TP), i.e., $\mathrm{Tr}_B \Phi(X) = \mathrm{Tr}_A X \ \forall\, X \in \mathcal{B}(\mathcal{H}_A)$, if and only if $\gamma_A = \frac{1}{d_A} I_A$.

When M_d or $\mathcal{B}(\mathcal{H})$ is equipped with the Hilbert–Schmidt inner product, one can define the adjoint, or dual, of a map Φ. We denote this by $\widehat{\Phi}$ and observe that this is equivalent to

$$\mathrm{Tr}\, B^\dagger \Phi(A) = \mathrm{Tr} [\widehat{\Phi}(B)]^\dagger A. \tag{5}$$

A matrix Φ is TP if and only if its adjoint $\widehat{\Phi}$ is unital.

It is a consequence of theorem 5 in [7] that the extreme points[7] of the convex set of CP maps for which $\gamma_A = \widehat{\Phi}(I_B) = \rho$ have a state representative (often called the Choi matrix) with rank $\leqslant$ rank ρ. We prefer to consider CPT maps and regard the density matrices with rank $\leqslant d_A$ as an extension of the set of extreme points. As shown in appendix B, this corresponds to the closure of the set of extreme points. We let $\mathcal{D}_C$ denote the set of density matrices in $\mathcal{B}(\mathcal{H}_C)$ or M_{d_C} and $\mathcal{D}_C(r)$ denote the subset of rank r. We also define the following subsets of $\mathcal{D}_{AB}(r)$:

$$\mathcal{P}_A(\rho; r, s) \equiv \{\gamma_{AB} \in \mathcal{D}_{AB} : \mathrm{rank}\, \gamma_{AB} = r, \mathrm{rank}\, \gamma_A = s \text{ and } \gamma_A = \rho\}, \tag{6a}$$

$$\mathcal{P}_A(r, s) \equiv \{\gamma_{AB} \in \mathcal{D}_{AB} : \mathrm{rank}\, \gamma_{AB} = r \text{ and } \mathrm{rank}\, \gamma_A = s\}. \tag{6b}$$

Although the sets in (6) above are subsets of $\mathcal{D}_{AB} \subset \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) \simeq M_{d_A} \otimes M_{d_B}$ we use the subscript A to emphasize that we impose conditions only on the marginal γ_A. When rank $\rho_1 =$ rank $\rho_2 = d_A$, the map

$$\gamma_{AB} \mapsto \left(\rho_2^{1/2} \rho_1^{-1/2} \otimes I_B\right) \gamma_{AB} \left(\rho_1^{-1/2} \rho_2^{1/2} \otimes I_B\right) \tag{7}$$

gives an isomorphism from $\mathcal{P}_A(\rho_1; r, d_A)$ to $\mathcal{P}_A(\rho_2; r, d_A)$ and each of these is isomorphic to $\mathcal{P}_A\left(\frac{1}{d_A} I_A; d_A, d_A\right)$ which is isomorphic to the set of CPT maps Φ whose Choi matrix has rank d_A. We will let $\mathcal{S}_A(\rho; r, s)$, etc denote the corresponding subsets of separable states in (6).

It will be useful to introduce the notation Υ_T for the map that takes a density matrix $\rho \mapsto T^\dagger \rho T$.

7 Choi's condition for true extreme points is implicit in theorem 1.4.6 of [1].

2. Maps with qubit inputs

2.1. Canonical form and parameterization

Now consider the case of CPT maps on qubits for which $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}_2$. As observed in [14], these maps can be written using the Bloch sphere representation in the form

$$\Phi\left(w_0 I + \sum_k w_k \sigma_k\right) = w_0 I + \sum_k (t_k w_0 + \lambda_k w_k)\sigma_k, \tag{8}$$

where σ_k denote the three Pauli matrices. Necessary and sufficient conditions on t_k, λ_k which ensure that Φ is CP are given in [18]. The form (8) is equivalent to representing Φ by a matrix T with elements $t_{jk} = \frac{1}{2}\mathrm{Tr}\, \sigma_j \Phi(\sigma_k)$ so that, with subscripts $j, k = 0, 1, 2, 3$ and the convention $I_2 = \sigma_0$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_2 & 0 & 0 & \lambda_3 \end{pmatrix}. \tag{9}$$

As shown in [14, appendix B] an arbitrary unital map on qubits can be reduced to this form by applying a variant of the singular value decomposition to the 3×3 submatrix with $j, k \in \{1, 2, 3\}$ using only real orthogonal rotations. Given the isomorphism between rotations and 2×2 unitary matrices, this corresponds to making a change of basis on the input and output spaces $\mathcal{H}_A = \mathbf{C}_{d_A} = \mathbf{C}_2$ and $\mathcal{H}_B = \mathbf{C}_{d_B} = \mathbf{C}_2$, respectively. Thus, for an arbitrary unital CP map Φ one can find unitary U, V such that $\Upsilon_{V^\dagger} \circ \Phi \circ \Upsilon_U$ has the form (8) or, equivalently, a matrix representative of the form (9).

It was shown in [18] that the maps with Choi rank $\leqslant 2$ are precisely those for which the form (9) becomes

$$T_{u,v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix} \tag{10}$$

with[8] u, v in $(-\pi, \pi] \times [0, \pi]$. Moreover, as shown in [16], the entanglement breaking (EB) maps are precisely the channels which have either $\cos u = 0$ or $\cos v = 0$.

It follows from (10) that every element of $\mathcal{P}_A\left(\frac{1}{2}I; 2, 2\right)$ can be represented by a triple $((u, v), U, V)$ consisting of a point in $\mathbf{R}_2$, and two unitary matrices U, V. However, some care must be taken so that each element of $\mathcal{P}_A\left(\frac{1}{2}I; 2, 2\right)$ is counted exactly once. It suffices to restrict (u, v) to the rectangle

$$\overline{\Delta} = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]. \tag{11}$$

Suitable rotations will give all allowed negative values of the non-zero elements in (10), as well as even permutations of t_k and λ_k. Problems with overcounting occur only on the lines $u = 0, v = 0, u = v$. To deal with this we define

$$\Delta = \left\{(u, v) : 0 < u \leqslant \frac{\pi}{2}, 0 < v \leqslant \frac{\pi}{2}, u \neq v\right\}. \tag{12}$$

(The line segments on the boundary with $u = \frac{\pi}{2}$ and $v = \frac{\pi}{2}$ are included in Δ as shown in figure 1.)

[8] The interval for u is shifted from that in [18]. However, the interval $[0, \pi]$ for v was incorrectly stated as $[0, \pi)$ in [18].

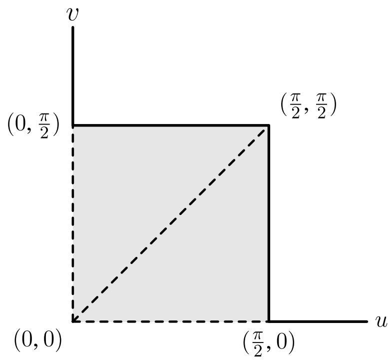

Figure 1. The rectangle $\overline{\Delta}$ corresponds to the shaded region. The dashed lines are not in Δ. The lines $u = \frac{\pi}{2}$ and $v = \frac{\pi}{2}$ correspond to the EB channels.

Because different pairs of matrices U, V may give the same channel on the lines not included in (12), we define equivalence classes as follows. Let $\mathcal{R}_t$ (with $t = x, y, z$) denote the subset of $SU(2)$ corresponding to the rotations around the indicated axis. We write $(U, V) \simeq (U', V')$ if there is an $R_t \in \mathcal{R}_t$ such that $U' = R_t U$ and $V' = R_t V$ or, equivalently $U'U^\dagger = V'V^\dagger \in \mathcal{R}_t$, and denote the quotient space $(SU(2) \times SU(2))/\mathcal{R}_t$. With this notation, we now make some observations:

(a) The subset of EB channels consists of those channels for which either $u = \frac{\pi}{2}$ or $v = \frac{\pi}{2}$.

(b) The line $u = v$ corresponds to the amplitude damping channels. (It is well known that only the case $u = v = \frac{\pi}{2}$ is EB; this is a completely noisy channel mapping to a fixed pure state.) From (10) one sees that these channels are invariant under rotations about the z-axis, and the set of amplitude damping channels in $\mathcal{P}_A\left(\frac{1}{2}I; 2, 2\right)$ is isomorphic to $(u, u) \times (SU(2) \times SU(2))/\mathcal{R}_z$.

(c) The line segments with $u = 0$ and $v = 0$ correspond to phase-damping channels. From (10) one sees that these channels are invariant under rotations about the x- and y-axes, respectively. Thus, the set of phase damping channels in $\mathcal{P}_A\left(\frac{1}{2}I; 2, 2\right)$ is isomorphic to

$$\left\{(0, v) : v \in \left(0, \frac{\pi}{2}, \right]\right\} \times (SU(2) \times SU(2))/\mathcal{R}_x$$
$$\bigcup \left\{(u, 0) : u \in \left(0, \frac{\pi}{2}\right]\right\} \times (SU(2) \times SU(2))/\mathcal{R}_y.$$

(d) The point $u = v = 0$ gives the identity channel, for which rank $\gamma_{AB} = 1$.

Thus $\mathcal{P}_A\left(\frac{1}{2}I; 2, 2\right)$ is isomorphic to

$$\begin{aligned} &\Delta \times SU(2) \times SU(2) \bigcup \{(u, u)\}_{u \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_z \\ &\bigcup \{(u, 0)\}_{u \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_y \\ &\bigcup \{(0, v)\}_{v \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_x \end{aligned} \tag{13}$$

and, $\mathcal{S}_A\left(\frac{1}{2}I; 2, 2\right)$, the subset of EB channels in $\mathcal{P}_A\left(\frac{1}{2}I; 2, 2\right)$, is isomorphic to

$$\begin{aligned} &\left\{\left(u, \frac{\pi}{2}\right)\right\}_{\{u \in (0, \frac{\pi}{2})\}} \times SU(2) \times SU(2) \\ &\bigcup \left\{\left(\frac{\pi}{2}, v\right)\right\}_{\{v \in (0, \frac{\pi}{2})\}} \times SU(2) \times SU(2) \bigcup \left(0, \frac{\pi}{2}\right) \times (SU(2) \times SU(2))/\mathcal{R}_x \\ &\bigcup \left(\frac{\pi}{2}, 0\right) \times (SU(2) \times SU(2))/\mathcal{R}_y \bigcup \left(\frac{\pi}{2}, \frac{\pi}{2}\right) \times (SU(2) \times SU(2))/\mathcal{R}_z. \end{aligned} \tag{14}$$

2.2. Construction of a measure

Let m_2 be the normalized Lebegue measure on $\overline{\Delta}$ and ν_2 the normalized Haar measure on $SU(2)$. Then the product measure $\widetilde{\mu} \equiv m_2 \times \nu_2 \times \nu_2$ defines a probability measure on $\Omega_2 = \overline{\Delta} \times SU(2) \times SU(2)$. Although every point in Ω_2 corresponds to an element in $\mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)$, it can happen, as described above, that more than one point corresponds to the same CPT map Φ. Therefore, to define a measure on $\mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)$ we use the map $g : \Omega_2 \to \mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)$ which takes

$$((u, v), U, V) \mapsto \Upsilon_V \circ \Phi_{u,v} \circ \Upsilon_{U^\dagger}, \tag{15}$$

where $\Phi_{u,v}$ denotes the CPT map whose Choi matrix is given by (10). The map g is surjective which allows us to define a measure μ on all sets $X \subset \mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)$ for which $g^{-1}(X)$ is measurable by

$$\mu(X) = \widetilde{\mu}(g^{-1}(X)). \tag{16}$$

Since g is surjective, $g^{-1}\big(\mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)\big) = \Omega_2$ which implies that $\mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)$ is measurable and $\mu\big(\mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big) = 1$. Thus, μ is a probability measure on $\mathcal{P}_A\big(\frac{1}{2}I; 2, 2\big)$.

Moreover, the entanglement breaking channels satisfy

$$\begin{aligned}\mu\left(\mathcal{S}_A\left(\frac{1}{2}I; 2, 2\right)\right) &= \widetilde{\mu}\left(\left\{\left(u, \frac{\pi}{2}\right) : u \in \left[0, \frac{\pi}{2}\right]\right\} \times SU(2) \times SU(2)\right.\\ &\quad \left.\bigcup \left\{\left(\frac{\pi}{2}, v\right) : v \in \left[0, \frac{\pi}{2}\right\} \times SU(2) \times SU(2)\right)\\ &= 0 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 1 = 0.\end{aligned} \tag{17}$$

Thus, we have proved the following.

Theorem 1. *A CPT map $\Phi : M_2 \mapsto M_2$ of Choi-rank 2 is almost surely not EB, or, equivalently, a state γ_{AB} on $\mathbf{C}_2 \otimes \mathbf{C}_2$ which has rank 2 and $\gamma_A = \frac{1}{2}I$ is almost surely entangled.*

Since the unitary conjugations have Choi matrices of rank 1, and correspond to the set $(0, 0) \times SU(2)$ which has measure zero, we have also proved the following result, which we state for completeness.

Theorem 2. *A CPT map $\Phi : M_2 \mapsto M_2$ of Choi-rank $\leqslant 2$ is almost surely not EB, or, equivalently, a state γ_{AB} on $\mathbf{C}_2 \otimes \mathbf{C}_2$ which has rank $\leqslant 2$ and $\gamma_A = \frac{1}{2}I$ is almost surely entangled.*

2.3. Removing the TP restriction

We would like to extend the results of the previous section to

Theorem 3. *If a state γ_{AB} on $\mathbf{C}_2 \otimes \mathbf{C}_2$ has rank 2 and γ_A also has rank 2, then γ_{AB} is almost surely entangled.*

Proof. As observed after (7), $\mathcal{P}_A(\rho_1; r, d_A) \simeq \mathcal{P}_A(\rho_2; r, d_A)$. Indeed, the CP maps corresponding to states in $\mathcal{D}_A(\rho; 2, 2)$ have the form $\Phi \circ \Upsilon_{\sqrt{2\rho}}$ with Φ CPT, although it might seem more natural to consider the dual $\Upsilon_{\sqrt{2\rho}} \circ \widehat{\Phi}$ which takes $I \mapsto d_A\rho$. Next, observe that any density matrix $\rho \in M_{d_A}$ of rank 2 can be written as $U\begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}U^\dagger$ with $x \in \big(0, \frac{1}{2}\big)$

and $U \in SU(2)$; the case $x = \frac{1}{2}$ gives $\frac{1}{2}I$ independent of U. Thus the set of density matrices $\rho \in M_{d_A}$ of rank 2 is isomorphic[9] to

$$\tfrac{1}{2}I \cup \left(0, \tfrac{1}{2}\right) \times SU(2) \tag{18}$$

and the set of bipartite density matrices $\mathcal{P}_A(2, 2)$ (for which rank γ_{AB} = rank γ_A = 2) is isomorphic to

$$\mathcal{P}_A\left(\tfrac{1}{2}I; 2, 2\right) \bigcup \mathcal{P}_A\left(\tfrac{1}{2}I; 2, 2\right) \times \left(0, \tfrac{1}{2}\right) \times SU(2). \tag{19}$$

To define a measure on this set, let m_1 denote the normalized Lebesgue measure on $\left(0, \frac{1}{2}\right)$ and let $\lambda_{2,t}$ be defined using the product measure so that

$$\lambda_{2,t}(X) = \begin{cases} t(\mu \times m_1 \times \nu_2)(X) & X \in \mathcal{P}_A\left(\tfrac{1}{2}I; 2, 2\right) \times \left(0, \tfrac{1}{2}\right) \times SU(2) \\ (1-t)\mu(X) & X \in \mathcal{P}_A\left(\tfrac{1}{2}I; 2, 2\right), \end{cases} \tag{20}$$

where we can pick any $t \in (0, 1]$ and μ is the measure defined in section 2.2. Then the subset of EB channels $\mathcal{S}_A(2, 2)$ has measure

$$\begin{aligned} \lambda_{2,t}(\mathcal{S}_A(2, 2)) &= \mu\left(\mathcal{S}_A\left(\tfrac{1}{2}I; 2, 2\right)\right) + \mu\left(\mathcal{S}_A\left(\tfrac{1}{2}I; 2, 2\right)\right) m_1\left(0, \tfrac{1}{2}\right) \nu_2(SU(2)) \\ &= t \cdot 0 + (1-t) \cdot 0 \cdot 1 \cdot 1 = 0 \end{aligned} \tag{21}$$

independent of $t \in (0, 1]$). We can drop the requirement that γ_A has rank 2 by observing that extension to all γ_{AB} of rank 2 requires only that one replaces $\left(0, \frac{1}{2}\right)$ on the right-hand side of (19) by $\left[0, \frac{1}{2}\right)$. Thus, we can conclude that

Corollary 4. *If a state γ_{AB} on $\mathbf{C}_2 \otimes \mathbf{C}_2$ has rank 2, then γ_{AB} is almost surely entangled.*

2.4. Two-dimensional subspaces of $\mathbf{C}_d$

We can use the isomorphism between $\mathbf{C}_2$ and any Hilbert space of dimension 2 to replace either $\mathcal{H}_A$ or $\mathcal{H}_B$ by a two-dimensional subspace of $\mathbf{C}_d$. However, for later use, we now want to extend our qubit results to the somewhat more general situation of the set of all CPT maps $\Phi : \mathbf{C}_2 \mapsto \mathbf{C}_{d_B}$ whose range has the form $\mathcal{B}(\text{span}\{|v_1\rangle, |v_2\rangle\})$ with $|v_1\rangle, |v_2\rangle \in \mathbf{C}_{d_B}$. Here, we do not fix the range, but consider all CPT maps whose range corresponds to some two-dimensional subspace of $\mathbf{C}_{d_B}$.

Observe that in the polar decomposition $\Upsilon_{V^\dagger} \circ \Phi \circ \Upsilon_U$ leading to the canonical form (8) we need only replace V by an isometry $V : \mathbf{C}_2 \mapsto \mathcal{H}_B$. Then in (13) and (14), the first use of $SU(2)$ in each subset must be replaced by $\mathcal{V}_d$ which is defined as the subset of $d \times 2$ matrices satisfying $V^\dagger V = I_2$. By theorem A.2 of [2], $\mathcal{V}_d$ can be given the structure of a real analytic manifold with a probability measure υ_d (which is unique if it is required to be left-invariant under $SU(d)$). Although $\mathcal{V}$ is not a group, we can define equivalence classes as before with $(V, U) \simeq (V', U')$ if there is a $R_t \in \mathcal{R}_t$ such that $V' = VR_t$ and $U' = UR_t$. Then the previous arguments go through with $SU(2) \times SU(2)$ replaced by $\mathcal{V}_d \times SU(2)$ in section 2.1 and the corresponding use of ν_2 in section 2.2 by υ_d.

[9] Here we use the fact that $\sigma_x \rho \sigma_x$ exchanges the eigenvalues. This is quite different from the situation in (12) where we could not assume $u < v$ because the permutation in S_3 which exchanges $1 \leftrightarrow 2$ cannot be implemented with a rotation.

3. General maps

3.1. CPT maps with $d_A > 2$

We now assume $d_A \geqslant d_B \geqslant 2$ and extend these results to bipartite states on $\mathbf{C}_{d_A} \otimes \mathbf{C}_{d_B}$ with $\gamma_A = \frac{1}{d_A} I_A$. We begin by considering a CPT map $\Phi : M_{d_A} \mapsto M_{d_B}$ with Choi-rank d_A. By theorem 5C of [11], which is equivalent to corollary 14, Φ can always be written in the form

$$\Phi(\rho) = \sum_k |g_k\rangle\langle g_k|\langle\psi_k, \rho\psi_k\rangle \tag{22}$$

where $\{g_k\}$ is an orthonormal basis for $\mathbf{C}_{d_A}$, but the states $\psi_k \in \mathbf{C}_{d_B}$ need *not* be orthogonal or even linearly independent. In the basis g_k, the Choi matrix for Φ has the form

$$\gamma_{AB} = \frac{1}{d_A} \sum_k |g_k\rangle\langle g_k| \otimes |\psi_k\rangle\langle\psi_k|, \tag{23}$$

which implies that γ_{AB} is block diagonal with each block a $d_B \times d_B$ rank-one projection. Let us first assume that ψ_1 and ψ_2 are linearly independent.

Now let $P_k \equiv |\psi_k\rangle\langle\psi_k|$ and write (23) explicitly in a block form as

$$\gamma_{AB} = \frac{1}{d_A} \begin{pmatrix} P_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & P_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & P_3 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & P_{d_A} \end{pmatrix} \tag{24}$$

and consider a density matrix of the form

$$\frac{1}{d_A} \begin{pmatrix} Q & 0 & 0 & \dots & 0 \\ 0 & P_3 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & \dots & P_{d_A} \end{pmatrix} \tag{25}$$

where $Q \in M_2 \otimes M_{d_B}$ is a positive semi-definite $2d_B \times 2d_B$ matrix of rank 2 satisfying $\mathrm{Tr}_B\, Q = I_2$. Now a density matrix of the form (25) is separable if and only if $\frac{1}{2} Q$ is separable. However, $\frac{1}{2} Q$ is a density matrix of the form considered in section 2.4.

Let $\mathcal{Y}_{d_A}(\{g_k\}, \{\psi_k\})$ denote the subset of $\mathcal{P}_A\big(\frac{1}{d_A} I_A; d_A, d_A, \big)$ consisting of density matrices of the form (25) or, equivalently,

$$\mathcal{Y}_{d_A}(\{g_k\}, \{\psi_k\}) = \left\{ Q \oplus \sum_{k=3}^{d_A} |g_k\rangle\langle g_k| \otimes |\psi_k\rangle\langle\psi_k| : Q \in \mathcal{X}_{AB}, \mathrm{Tr}_B\, Q = I_2 \right\}, \tag{26}$$

where $\oplus$ denotes the direct sum and

$$\mathcal{X}_{AB} = \mathcal{B}(\mathrm{span}\{|g_1\rangle, |g_2\rangle\} \otimes \mathcal{B}(\mathrm{span}\{|\psi_1\rangle, |\psi_2\rangle\}). \tag{27}$$

The set of projections $|\psi_k\rangle\langle\psi_k| \in M_{d_B}$ is isomorphic to S_{2d_B-1}, the ℓ_2 unit sphere in $\mathbf{R}_{2d_B}$. For a given $|g_1\rangle, |g_2\rangle$, the set $\mathcal{X}_{AB}$ depends only on span $\{|\psi_1\rangle, |\psi_2\rangle\}$ and not the choice of individual vectors. Therefore, we can identify each point in

$$\begin{aligned} \Omega_{d_A} &\equiv \overline{\Delta}_2 \times \mathcal{V}_{d_B} \times SU(2) \times SU(d_A)/SU(2) \times \underbrace{S_{2d_B-1} \times \ldots \times S_{2d_B-1}}_{d_A-2} \\ &= \overline{\Delta}_2 \times \mathcal{V}_{d_B} \times SU(d_A) \times \underbrace{S_{2d_B-1} \times \cdots \times S_{2d_B-1}}_{d_A-2} \end{aligned} \tag{28}$$

with a density matrix γ_{AB} in $\mathcal{Y}_{d_A} \equiv \bigcup_{\{g_k\},\{\psi_k\}} \mathcal{Y}_{d_A}(\{g_k\}, \{\psi_k\})$, the set of all density matrices of the form (25). (Note that S_{2d_B-1} occurs $d_A - 2$ times in (28) corresponding to the choices of ψ_k for $k = 3, 4, \ldots, n$. The set $\mathcal{Y}_{d_A}(\{g_k\}, \{\psi_k\})$ depends only on span $\{|\psi_1\rangle, |\psi_2\rangle\}$ = range V with $V \in \mathcal{V}_{d_B}$, with non-orthogonal vectors $|\psi_1\rangle, |\psi_2\rangle$ associated with non-unital qubit channels via isomorphism.)

Let m_2 and v_d be measures as in sections 2.2 and 2.4, let ν_d be the normalized Haar measure on $SU(d)$ and let n_{2d_B-1} be a probability measure on S_{2d_B-1}. We define a normalized measure $\widetilde{\mu}$ on Ω_{d_A} by the product measure

$$\widetilde{\mu} = m_2 \times v_{d_B} \times \nu_2 \times \nu_{d_A/2} \times \underbrace{n_{2d_B-1} \times \cdots \times n_{2d_B-1}}_{d_A-2}. \tag{29}$$

To obtain a measure on $\mathcal{Y}_{d_A}$ we proceed as in section 2.2. Let $G : \Omega_{d_A} \mapsto \mathcal{Y}_{d_A}$ be the map that sends an element $\big((u, v), V, U, |\psi_3\rangle, \ldots, |\psi_{d_A}\rangle\big)$ to the corresponding density matrix in $\mathcal{Y}_{d_A}$ and define

$$\mu(X) = \widetilde{\mu}(G^{-1}(X)) \tag{30}$$

whenever $X \subset \mathcal{Y}_{d_A}$ for which $G^{-1}(X)$ is measurable.

As explained above, corollary 14 implies that $\mathcal{S}_A\big(\frac{1}{d_A} I_A; d_A, d_A\big) \subset \mathcal{Y}_{d_A}$. Then, proceeding as in (17), one finds

$$\mu\left(\mathcal{S}_A\left(\frac{1}{d_A} I_A; d_A, d_A\right)\right) = 0 \cdot 1 \cdot 1 \cdot 1^{d_A-2} = 0. \tag{31}$$

Moreover, for any reasonable extension of μ from $\mathcal{Y}_{d_A}$ to all of $\mathcal{P}_A\big(\frac{1}{d_A} I_A; d_A, d_A\big)$, the EB subset will still have measure zero. In particular, one could simply let

$$\omega(X) = \begin{cases} \mu(X) & \text{if} \quad X \subset \mathcal{Y}_{d_A} \\ 0 & \text{if} \quad X \subset \mathcal{P}_A\left(\dfrac{1}{d_A} I_d; d_A, d_A\right) \Big\backslash \mathcal{Y}_{d_A} \end{cases} \tag{32}$$

and note that ω is absolutely continuous with respect to any other extension of μ.

Thus, we have reduced the general case to that of $d_A = 2$ and conclude that

Theorem 5. *Let γ_{AB} be a state on $\mathbf{C}_{d_A} \otimes \mathbf{C}_{d_B}$ which has rank $d_A \geqslant d_B \geqslant 2$ and for which $\gamma_A = \frac{1}{d_A} I_A$. Then γ_{AB} is almost surely entangled.*

Remark. The assumption that ψ_1 and ψ_2 are linearly independent can be dropped because that case corresponds to $u = v = \frac{\pi}{2}$ in (8) and is included implicitly in our analysis. The set of channels for which all ψ_j are identical also has measure zero, except for the excluded situation $d_B = 1$, for which all states are separable.

3.2. Reduction of the general case to CPT

As observed earlier, when rank $\rho = d_A$

$$\mathcal{P}_A(\rho; d_A, d_A) = \left\{ \big(\sqrt{d_A \rho} \otimes I_B\big) \gamma_{AB} \big(\sqrt{d_A \rho} \otimes I_B\big) : \gamma_{AB} \in \mathcal{P}_A\left(\frac{1}{d_A} I_A; d_A, d_A\right)\right\} \tag{33}$$

is isomorphic to $\mathcal{P}_A\big(\frac{1}{d_A} I_A; d_A, d_A\big)$. But parameterizing the set of density matrices of rank d_A is a bit more subtle than for $d_A = 2$ because of the need to consider degenerate eigenvalues, for situations beyond $\frac{1}{d_A} I$. However, this only affects a set of measure zero and can be dealt with as in the preceding sections. To describe the set of density matrices of rank d_A consider the set of vectors

$$Z = \left\{ \mathbf{z} = (\zeta_1, \zeta_2, \dots \zeta_{d_A}) : 0 < \zeta_1 \leqslant \zeta_2 \leqslant \cdots \leqslant \zeta_{d_A}, \sum_k \zeta_k = 1 \right\} \qquad (34)$$

in the positive facet of the ℓ_1 unit ball of $\mathbf{R}_{d_A}$. We can associate each $\mathbf{z} \in Z$ with a diagonal matrix $\Lambda_{\mathbf{z}}$ so that that the map $h : (\mathbf{z}, U) \mapsto U\Lambda_{\mathbf{z}}U^\dagger$ takes $Z \times SU(d_A)$ onto $\mathcal{D}_A(d_A)$, the set of density matrices in M_{d_A} with full rank d_A. Since we can identify Z with a subset of $\mathbf{R}_{d_A-1}$, we put normalized Lebesgue measure m_{d_A-1} on Z, and let

$$\eta_{d_A}(X) = (m_{d_A-1} \times \nu_{d_A})(h^{-1}(X)) \qquad (35)$$

whenever $X \subset \mathcal{D}_A(d_A)$ and $h^{-1}(X)$ is measurable. Then it follows from (31) that for any extension ω of μ, the product measure $\omega \times \eta_{d_A}$ gives a measure on $\mathcal{P}_A(d_A, d_A)$ for which the separable states $\mathcal{S}_A(d_A, d_A)$ have measure $0 \cdot 1 = 0$. Thus, we have proved

Theorem 6. *If a state γ_{AB} on $\mathbf{C}_{d_A} \otimes \mathbf{C}_{d_B}$ has rank $d_A \geqslant d_B \geqslant 2$ and rank$(\gamma_A) = d_A$ then γ_{AB} is almost surely entangled.*

3.3. Further results

Theorem 11 states that if the rank of γ_A is d_A and the rank of γ_{AB} is strictly smaller than d_A, then γ_{AB} is entangled. Thus $r < d_A$ implies that $\mathcal{P}_A(\rho; r, d_A)$ consists entirely of entangled states. If we combine this with our results for $r = s = d_A$ we obtain several additional theorems, which we state for completeness.

Theorem 7. *Assume $d_A \geqslant d_B \geqslant 2$. If a state γ_{AB} on $M_{d_A} \otimes M_{d_B}$ has rank $\gamma_{AB} \leqslant d_A =$ rank γ_A, then γ_{AB} is almost surely entangled.*

By using the isomorphism between $\mathbf{C_d}$ and any Hilbert space of dimension d we can restate this by letting $\mathcal{H}_A =$ range γ_A and $\mathcal{H}_B =$ range γ_B and considering γ_{AB} as a state on $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_A)$.

Theorem 8. *If a state γ_{AB} on $M_{d_A} \otimes M_{d_B}$ has rank $\gamma_{AB} \leqslant$ rank γ_A and rank $\gamma_A \geqslant$ rank $\gamma_B \geqslant 2$, then γ_{AB} is almost surely entangled.*

We also find that we can eliminate the need to consider the rank of γ_A.

Theorem 9. *Assume $d_A \geqslant d_B \geqslant 2$. If a state γ_{AB} on $M_{d_A} \otimes M_{d_B}$ has rank $\gamma_{AB} \leqslant d_A$, then γ_{AB} is almost surely entangled.*

Proof. Let $\overline{Z}$ denote the closure of (34). Since this simply replaces the strict inequality $0 < \zeta_1$ by $0 \leqslant \zeta_1$, the set $\overline{Z} \times SU(d)$ includes all density matrices in M_{d_A} so that

$$\overline{Z} \backslash Z \times SU(2) = h^{-1}(\{\rho \in \mathcal{P}_A : \text{rank } \rho < d_A\}). \qquad (36)$$

Now extend the measure η in (35) to all of $\mathcal{D}_A$. The set of all separable states γ_{AB} with rank $\gamma_{AB} \leqslant d_A$ is $\mathcal{S}_A(d_A) \equiv \bigcup_{s \leqslant d_A} \mathcal{S}_{d_A}(d_A, s)$. The subset of separable states with rank $\gamma_A < d_A$ satisfies

$$\bigcup_{s<d_A} \mathcal{S}_{d_A}(d_A, s) \subset \{\rho \in \mathcal{P}_A : \text{rank } \rho < d_A\}. \qquad (37)$$

But

$$\eta_{d_A}(\{\rho \in \mathcal{P}_A : \text{rank } \rho < d_A\}) = m_{d_A-1}(\overline{Z} \backslash Z)\nu_{d_A}(SU(2)) = 0 \cdot 1. \qquad (38)$$

Thus

$$(\omega_{d_A} \times \eta_{d_A})(\mathcal{S}(d_A)) = (\omega_{d_A} \times \eta_{d_A})(\mathcal{S}_{d_A}(d_A, d_A)) + (\omega_{d_A} \times \eta_{d_A})\left(\bigcup_{s<d_A} \mathcal{S}_{d_A}(d_A, s)\right)$$
$$\leqslant 0 \cdot 1 + 1 \cdot 0 = 0. \tag{39}$$

□

4. Final comments

4.1. Remarks on measure

If we apply the argument used to prove theorem 9 to the subset of states with $\gamma_A = \frac{1}{d_A} I_{d_A}$ or equivalently, combine theorems 5 and 11 we obtain the following result which we state in terms of channels.

Corollary 10. *Assume $d_A \geqslant d_B \geqslant 2$. Then the set of CPT maps $\Phi : \mathbf{C}_{d_A} \mapsto \mathbf{C}_{d_B}$ whose Choi matrix has rank $r \leqslant d_A$ is almost surely entanglement breaking.*

As shown in appendix B, the closure of the set of extreme points of CPT maps $\Phi : \mathbf{C}_{d_A} \mapsto \mathbf{C}_{d_B}$ is precisely the set of channels whose Choi matrix has rank $\leqslant d_A$. Because the extreme points of a convex set lie on the boundary, their closure always has measure zero. Thus, corollary 10 is a special case of a well-known, more general fact from convex geometry. An alternative, and somewhat simpler, approach to proving theorem 6 would be to use this fact together with theorem 15. However, we feel that it is useful to see the specific paramaterizations which lead to our results. In our approach, one sees that everything really follows from the basic paramaterization of extreme points for qubit channels, and the fact that (up to sets of measure zero) the relevant sets of bipartite states can be parameterized as direct products on which we can put product measures.

One could extend corollary 10 to the set of CP maps for which $\widehat{\Phi}(I_B) = \rho$ with $\rho \in \mathcal{D}_A(r)$ fixed, again using the fact that the closure of the set of extreme points has measure zero. Then we can conclude that the subset of separable states $\cup_{s\leqslant r}\mathcal{S}(\rho; r, s)$ has measure zero with respect to a measure on $\cup_{s\leqslant r}\mathcal{P}(\rho; r, s)$. However, we cannot go directly from this observation to theorem 9 by taking the $\bigcup_{\rho\in\mathcal{D}_A}$ because the set $\mathcal{D}_A$ is uncountable. One would still need the argument in section 3.2. What this observation about extreme points does tell us is that our results are not sensitive to the choice of measure. The fundamental issue is that the bipartite states can be parameterized as a smooth manifold on which the separable ones correspond to a space of smaller dimensions.

There is one unsatisfying aspect of using the inverse image to define a measure, as in (16); namely, that it does not reflect the fact that different unitaries give the same map on some lines in $\overline{\Delta}$. An alternative would be to first define separate measures on the different regions in (13), e.g., on $\{(u, u)\}_{u\in(0,\frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_z$, use the product measure $m_1 \times \widetilde{\nu}_z$ where m_1 is a normalized Lebesque measure on $\left(0, \frac{\pi}{2}\right)$ and $\widetilde{\nu}_z$ is Haar measure on the group $(SU(2) \times SU(2))/\mathcal{R}_z$. One could then combine the measures on the four subsets in (13) as in (20) using, say, weights $1 - t_x - t_y - t_z, t_z, t_x, t_y$ with $t_m \geqslant 0$ and $\sum_{m=1}^{3} t_m \leqslant 1$. However, given that each of the line segments with $u = \frac{\pi}{2}$, $u = v$ and $v = \frac{\pi}{2}$ has measure zero in $\overline{\Delta}$, the most natural choice weight would be $t_m = 0$, equivalent to simply omitting the corresponding channels (or states).

In fact, all situations in which a quotient space is needed, as in sections 2.2 and 3.2 have measure zero in our inverse image approach. Intuitively, one would like to simply

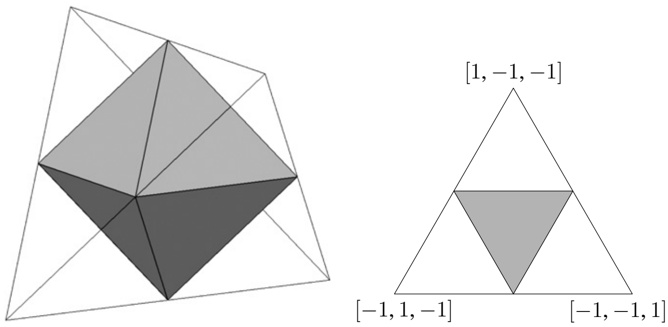

Figure 2. The left figure shows the tetrahedron of unital qubit channels with the octahedron of the EB subset. The right figure shows one of the faces of the tetrahedron, corresponding to channels with Choi-rank 3, with the shaded region as the subset of EB channels.

observe that we can identify $\mathcal{D}_A(\frac{1}{2}I; 2, 2)$ with a subset of $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$ that satisfies $\Delta \subset \mathcal{D}_A\left(\frac{1}{2}I; 2, 2\right) \subset \overline{\Delta}$ and then observe that since

$$\mu(\Delta) \leqslant \mu\left(\mathcal{D}_A\left(\tfrac{1}{2}I; 2, 2\right)\right) \leqslant \mu(\overline{\Delta}) \tag{40}$$

and $\mu(\Delta) = \mu(\overline{\Delta}) = 1$, one must have $\mu\left(\mathcal{D}_A\left(\frac{1}{2}I; 2, 2\right) = 1\right.$. But to use this approach, one must establish that $\mathcal{D}_A\left(\frac{1}{2}I; 2, 2\right)$ can be identified with a measurable subset of $\overline{\Delta}$.

4.2. Optimality

It is natural to ask if the results in theorems 7 and 8 are optimal. For $d_A > 2$, it is clear that the results which follow from those of Walgate and Scott [19] are better. Thus, the question becomes whether or not rank $\gamma_{AB} \leqslant (d_A - 1)(d_B - 1)$ is optimal. This does not follow from the subspace theorems in [19] because when rank $\gamma_{AB} = 2 > (d_A - 1)(d_B - 1)$ the product states can form a set of measure zero in a subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$. However, we know that the separable ball in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ has strictly positive measure [4, 5, 21] so that the optimal rank must be strictly smaller than $d_A d_B$.

In the case of qubits, we know that theorem 3 is stronger than the results implied by Walgate and Scott [19], and that when rank $\gamma_{AB} = 4$, the separable states have strictly positive measure. If we restrict attention to those states γ_{AB} with rank 3 and $\gamma_A = \gamma_B = \frac{1}{2}I$ or, equivalently, the unital CPT maps with Choi-rank 3, we can use the familiar picture of a tetrahedron [8, 16, 18]. The rank-3 states correspond to the faces, and the subset of separable states on each face to the smaller triangle whose vertices are midpoints of the edges is shown in figure 2. Thus, the unital CPT maps with Choi-rank 3 have measure 0.25 with respect to all the unital CPT maps on qubits. However, we do not know if something similar holds when the restriction to unital maps is removed. Thus, the question of whether $\mathcal{P}_A\left(\frac{1}{2}I; 3, 2\right)$ has measure zero or positive measure seems to be open.

Acknowledgments

It is a pleasure to thank Jonathon Walgate for bringing [19] to our attention, to Michael Wolf for providing [12] and to Michael Nathanson for assistance with figures.

Appendix A. Some separability theorems

For completeness, we now state and sketch proofs of some results that are well known and/or proved in [11]. The first result appeared as [12, theorem 1] in a slightly stronger form.

Theorem 11. *If rank $\gamma_{AB} < d_A =$ rank γ_A, then γ_{AB} is not separable.*

Proof. First observe that γ_{AB} is separable if and only if

$$\widetilde{\gamma}_{AB} \equiv \frac{1}{d_A}(I_A \otimes \gamma_A)^{-1/2}\gamma_{AB}(I_A \otimes \gamma_A)^{-1/2} \tag{A.1}$$

is separable. But $\widetilde{\gamma}_A = \frac{1}{d_A} I_A$. Now both the reduction and majorization criteria [6, 9] for separability of a state ρ_{AB} imply that the largest eigenvalue must satisfy $\|\rho_{AB}\|_\infty \leqslant \|\rho_B\|_\infty$. But rank $\widetilde{\gamma}_{AB} =$ rank $\gamma_{AB} < d_A$ implies that $\widetilde{\gamma}_{AB}$ has at least one eigenvalue $> \frac{1}{d_A}$. Thus $\|\widetilde{\gamma}_{AB}\|_\infty > \frac{1}{d_A} = \|\widetilde{\gamma}_B\|_\infty$, and it follows that both $\widetilde{\gamma}_{AB}$ and γ_{AB} are entangled. □

When rank $\gamma_A < d_A$, one can regard the underlying Hilbert space $\mathcal{H}_A$ to be range $\gamma_A = (\ker \gamma_A)^\perp$. One then obtains

Corollary 12. *If rank $\gamma_{AB} <$ rank γ_A, then γ_{AB} is not separable.*

The following lemma goes back at least to [10] and a simpler proof was given in [11]. To emphasize that one need not assume $d_A = d_B$ (and because of typos in [11]) we include a full proof here.

Lemma 13. *Let ρ_{AB} be a density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. If ρ_{AB} is separable, ρ_{AB} has rank d, and ρ_A has rank d, then ρ_{AB} can be written as a convex combination of products of pure states using at most d products.*

Proof. Since ρ_{AB} is separable it can be written in the form

$$\rho_{AB} = \sum_{i=1}^{k} \lambda_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|. \tag{A.2}$$

with $\|a_i\| = \|b_i\| = 1$. Assume that $k > d$ and that ρ_{AB} cannot be written in the form (A.2) using less than k products. Since ρ_A has exactly rank d, there is no loss of generality in assuming that the vectors above have been chosen so that $|a_1\rangle, |a_2\rangle, \ldots, |a_d\rangle$ are linearly independent. Moreover, since ρ_{AB} has rank $d < k$, the first $d + 1$ vectors $|a_i\rangle \otimes |b_i\rangle$ must be linearly dependent so that one can find α_j such that

$$\sum_{j=1}^{d+1} \alpha_j |a_j\rangle \otimes |b_j\rangle = 0. \tag{A.3}$$

Now let $\{|e_k\rangle\}$ be an orthonormal basis for $\mathcal{H}_B$. Then

$$\sum_{j=1}^{d+1} \alpha_j \langle e_k, b_j\rangle |a_j\rangle = 0 \;\forall\, k. \tag{A.4}$$

Since the first d vectors $|a_j\rangle$ are linearly independent, the solution of $\sum_j x_j |a_j\rangle = 0$ is unique up to a multiplicative constant. Applying this to the coefficients in (A.4) one finds that there are numbers ν_k such that $\alpha_j \langle e_k, |b_j\rangle = \nu_k x_j$. Let $|\nu\rangle \equiv \sum_k \nu_k |e_k\rangle$. Then $\alpha_j |b_j\rangle = x_j |\nu\rangle$. Since multiplying x_j by c changes $\nu_k \to \frac{1}{c}\nu_k$, one can assume that x_j has been chosen so that

$\|\nu\| = 1 = \|b_j\|$. Then $\alpha_j|b_j\rangle = x_j\, e^{i\theta_j}|\nu\rangle$, and $\alpha_j \neq 0$ implies $|b_j\rangle = e^{i\theta_j}|\nu\rangle$. Therefore, one can rewrite (A.2) as

$$\rho_{AB} = \sum_{j:\alpha_j=0} \lambda_j |a_j\rangle\langle a_j| \otimes |b_j\rangle\langle b_j| + \sum_{j:\alpha_j\neq 0} \lambda_j |a_j\rangle\langle a_j| \otimes |\nu\rangle\langle\nu|. \tag{A.5}$$

Suppose that t of the α_j are nonzero. Since the vectors $\{|a_j\rangle : \alpha_j \neq 0\}$ are linearly dependent, the density matrix $\sum_{j:\alpha_j\neq 0} \lambda_j|a_j\rangle\langle a_j|$ has rank strictly $< t$ and can be rewritten in the form $\sum_{k=1}^{s} \lambda'_j |a'_j\rangle\langle a'_j|$ using only $s < t$ vectors $|a'_j\rangle$. Substituting this into (A.5) gives ρ_{AB} as a linear combination of products using strictly less than k contradicting the assumption that (A.2) used the minimum number. □

Corollary 14. *If γ_{AB} is separable and $\gamma_A = \frac{1}{d_A} I_A$, then γ_{AB} can be written in the form*

$$\gamma_{AB} = \sum_k \frac{1}{d_A} |g_k\rangle\langle g_k| \otimes |\psi_k\rangle\langle\psi_k| \tag{A.6}$$

with g_k an orthonormal basis for $\mathbf{C}_{d_A}$.

Proof. Since γ_{AB} is separable it is a convex combination of projections onto product states and can be written in the form

$$\gamma_{AB} = \sum_k \xi_k |g_k \otimes \psi_k\rangle\langle g_k \otimes \psi_k|. \tag{A.7}$$

Since rank γ_A is d_A by assumption, it follows from the lemma in appendix A that we can assume $k = 1, 2, \ldots, d_A$ (duplicating terms if $< d_A$ are needed). But then, the assumption

$$\frac{1}{d_A} I_A = \gamma_A = \sum_k \xi_k |g_k\rangle\langle g_k| \tag{A.8}$$

holds if and only if $\xi_k = \frac{1}{d_A} \forall k$ and the vectors g_k are orthonormal. □

Appendix B. Closure of the set of extreme points

It is often useful to consider the set of all CPT maps with Choi rank $\leqslant d_A$. In [18] these were called generalized extreme points and shown to be equivalent to the closure of the set of extreme points for qubit maps. This is true in general[10]. We repeat here an argument from [17]. Let $\mathcal{E}(d_A, d_B)$ denote the extreme points of the convex set of CPT maps from M_{d_A} to M_{d_B}.

Theorem 15. *The closure $\overline{\mathcal{E}(d_A, d_B)}$ of the set of extreme points of CPT maps $\Phi : M_{d_A} \mapsto M_{d_B}$ is precisely the set of such maps with Choi rank at most d_A.*

Proof. Let A_k be the Choi–Kraus operators for a map $\Phi : M_{d_A} \mapsto M_{d_B}$ with Choi-rank $r \leqslant d_A$ which is not extreme, and let B_k be the Choi–Kraus operators for a true extreme point with Choi-rank d_A. When $r < d_A$ extend A_k by letting $A_m = 0$ for $m = r+1, r+2, \ldots, d_A$ and define $C_k(\epsilon) = A_k + \epsilon B_k$. There is a number ϵ_* such that the d_A^2 matrices $C_j^\dagger(\epsilon)C_k(\epsilon)$ are linearly independent for $0 < \epsilon < \epsilon_*$. To see this, for each $C_j^\dagger(\epsilon)C_k(\epsilon)$ 'stack' the columns to give a vector of length d_A^2 and let $M(\epsilon)$ denote the $d_A^2 \times d_A^2$ matrix formed with these vectors as columns. Then det $M(\epsilon)$ is a polynomial of degree d_A^4, which has at most d_A^4 distinct roots. Since the matrices $A_j^\dagger A_k$ were assumed to be linearly dependent, one of these roots is 0; it

[10] Arveson [3] has pointed out that theorem 15 can also be proved using results in [2].

suffices to take ϵ_* the next largest root (or +1 if no roots are positive). Thus, the operators $C_j^\dagger(\epsilon)C_k(\epsilon)$ are linearly independent for $\epsilon \in (0, \epsilon_*)$. The map $\rho \mapsto \sum_k C_k(\epsilon)\rho C_k^\dagger(\epsilon)$ is CP, with

$$\sum_k C_k^\dagger(\epsilon)C_k(\epsilon) = (1+\epsilon^2)I + \epsilon\big(A_k^\dagger B_k + B_k^\dagger A_k\big) \equiv S(\epsilon).$$

For sufficiently small ϵ the operator $S(\epsilon)$ is positive semi-definite and invertible, and the map $\Phi_\epsilon(\rho) = C_k(\epsilon)S(\epsilon)^{-1/2}\rho S(\epsilon)^{-1/2}C_k^\dagger(\epsilon)$ is a CPT map with Kraus operators $C_k(\epsilon)S(\epsilon)^{-1/2}$. Thus, one can find ϵ_c such that $\epsilon \in (0, \epsilon_c)$ implies that $\Phi_\epsilon \in \mathcal{E}(d_A, d_B)$. It then follows from $\lim_{\epsilon\to 0+} \Phi_\epsilon = \Phi$ that $\Phi \in \overline{\mathcal{E}(d_A, d_B)}$.

□

References

[1] Arveson W 1969 Subalgebras of C*-algebras *Acta Math.* **123** 141–224

[2] Arveson W 2009 The probability of entanglement *Commun. Math. Phys.* **286** 283–312 (arXiv:0712.4163)

[3] Arveson W Private correspondence

[4] Aubrun G and Szarek S J 2006 Tensor products of convex sets and the volume of separable states on *N* qudits *Phys. Rev.* A **73** 022109 (arXiv:quant-ph/0503221)

[5] Gurvits L and Barnum H 2002 Largest separable balls around the maximally mixed bipartite quantum state *Phys. Rev.* A **66** 062311 (arXiv:quant-ph/0204159)

[6] Bruss D 2002 Characterizing entanglement *J. Math. Phys.* **43** 4237 (arXiv:quant-ph/0110078)

[7] Choi M-D 1975 Completely positive linear maps on complex matrices *Linear Algebr. Appl.* **10** 285–90

[8] Horodecki R and Horodecki M 1996 Information-theoretic aspects of quantum inseparability of mixed states *Phys. Rev.* A **54** 1838–43 (arXiv:quant-ph/9607007)

[9] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2007 Quantum entanglement arXiv: quant-ph/0702225

[10] Horodecki P, Lewenstein M, Vidal G and Cirac I 2000 Operational criterion and constructive checks for the separability of low rank density matrices *Phys. Rev.* A **62** 032310 (arXiv:quant-ph/0002089)

[11] Horodecki M, Shor P and Ruskai M B 2003 Entanglement breaking channels *Rev. Math. Phys.* **15** 629–41 (arXiv:quant-ph/030203)

[12] Horodecki P, Smolin J A, Terhal B M and Thapliyal A V 2003 Rank two bipartite bound entangled states do not exist *Theor. Comput. Sci.* **292** 589–96 (arXiv:quant-ph/9910122)

[13] Jamiolkowski A 1972 Linear transformations which preserve trace and positive semidefiniteness of operators *Rep. Math. Phys.* **3** 275–8

[14] King C and Ruskai M B 2001 Minimal entropy of states emerging from noisy quantum channels *IEEE Trans. Inf. Theory* **47** 1–19 (arXiv:quant-ph/9911079)

[15] Parthasarathy K R 2004 On the maximal dimension of a completely entangled subspace for finite level quantum systems *Proc. Indian Acad. Sci.* **114** 365

[16] Ruskai M B 2003 Qubit entanglement breaking channels *Rev. Math. Phys.* **15** 643–62 (arXiv: quant-ph/0302032)

[17] Ruskai M B 2007 Some open problems in quantum information theory arXiv:0708.1902

[18] Ruskai M B, Szarek S and Werner E 2002 An analysis of completely positive trace-preserving maps M_2 *Linear Algebr. Appl.* **347** 159

[19] Walgate J and Scott A J 2008 Generic local distinguishability and completely entangled subspaces *J. Phys. A: Math. Theor.* **41** 375305 (arXiv:0709.4238)

[20] Wallach N 2002 An unentangled gleason's theorem *Contemp. Math.* **305** 291–7 (arXiv:quant-ph/0002058)

[21] Zyczkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 On the volume of the set of mixed entangled states *Phys. Rev.* A **58** 883–92 (arXiv:quant-ph/9804024)